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A Study on Natural Convection from Two Cylinders in a Cavity

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Steady-state natural convection heat transfer characteristics from cylinders in a multiply-connected bounded region are clarified. A spectral finite difference scheme (spectral decomposition of the system of partial differential equations, semi-implicit time integration) is applied in numerical analysis, with a boundary-fitted conformal coordinate system through a Jacobian elliptic function with a successive transformation to formulate a system of governing equations in terms of a stream function, vorticity and temperature. Multiplicity of the domain is expressed explicitly.

Key Words : Natural Convection, Spectral Finite Difference Scheme, Cavity, Numerical Analysis, Vorticity

1. Introduction

Methods applied in numerical analysis in the field of fluid dynamics are mainly divided into finite difference schemes and finite element schemes in a point of ways for formulation of a system of equations from the original differential equations supplemented with boundary conditions and/or initial conditions. The former (FDM) is mostly based on finite difference approximation for derivatives, whereas the latter (FEM) is based on a principle of weighted residuals. Conventional finite element method adopt polynomials as a base for expanding (or approximating) unknowns in a finite series.

On the other hand, spectral finite element methods, just called spectral methods, adopt a finite part of a complete set of periodic functions such as sine and cosine functions as a base. Thus FDM and FEM substantially involve errors such as truncation errors in deriving discretized equations. To avoid such errors in discretization, the introduced scheme is a spectral method mainly developed by one of the authors (Mochimaru, 1998), where the original governing equations with boundary conditions are exactly decomposed into each component of the given complete series such as Fourier series and Dini series. Under these formulation no errors are introduced as long as the series converges. However, the algorithm used by a current spectral finite difference scheme in this paper is completely different from the so-called spectral methods (Canuto et al., 1988 ; Boyd et al., 1989), supplemented with exact spectral decomposition.

The effectiveness of the current spectral finite difference scheme was shown for natural convection heat transfer from a circular cylinder in good agreement with experimental data (Mochimaru, 1988), for a flow past an elliptic cylinder in good

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agreement with experimental data (Mochimaru, 1990), for a steady flow behind a body of revolution in good agreement with experimental data (Mochimaru, 1992) for a simply-connected region. In an engineering point of view, we frequently meet multiply-connected region to be treated. In a mathematical point of view, the condition to the multiply-connectedness of the region should be supplemented explicitly for such a region, although in FDM and FEM such condition is not used explicitly, instead with additional discretization. One analysis in a triply-connected region using a spectral finite difference scheme is given (Mochimaru and Bae, 2003).

Although a steady-state asymptotic thermal plume behaviour from a cylinder in a sufficient but finite extension is established (Mochimaru, 2004), no steady-state two-dimensional natural convection from cylinder at an equal uniform surface temperature in an infinite extension of uniform temperature exists, whereas in case of different surface temperatures a steady-state solution is possible (Mochimaru and Bae, 2002a ; 2002b). In this paper, steady-state natural laminar convection heat transfer characteristics from two nearly circular cylinders placed in a cavity are clarified, using a spectral finite difference scheme.

2. Analysis

2.1 Configuration

A spectral finite difference method (Mochimaru, 1998) is applied for a steady-state laminar natural convection in a two-dimensional triply-connected region (x, y), where a boundary-fitted conformal mapping coordinate system (α, β), $|\beta| \leq \pi$ is formed in terms of a Jacobian elliptic function sn such that

$$\frac{\cosh(\alpha+i\beta)}{\cosh \alpha_\infty}=\frac{1+i\,\mathrm{sn}(\zeta^*, k)}{1-i\,\mathrm{sn}(\zeta^*, k)},\quad 0\leq\alpha\leq\alpha_\infty \tag{1}$$

$$\begin{aligned} &\zeta^*=i\left(z+\frac{c^2}{z-a}+\frac{d^2}{z-b}\right)\\ &z\equiv x+iy\,;\ 0<a<b\\ &c, d>0,\ c\ll 1,\ d\ll 1 \end{aligned} \tag{2}$$

k : modulus, given a priori. a, b, c, d are chosen so as to constitute two separate nearly circular cylinders for $-i\zeta^*>0$. In this configuration, centers of the two cylinders are located in the same elevation level $y=0$. The negative y-direction is assumed to be in the gravity direction. The cavity wall is specified by $\alpha=\alpha_\infty$, and the cylinder surfaces are given by $\alpha=0$ (a necessary condition). The four locations of the cylinder surface on the straight line $y=0$, z_i, $i=1, \cdots, 4$ are given by

$$1-\left(\frac{c}{z_i-a}\right)^2-\left(\frac{d}{z_i-b}\right)^2=0 \tag{3}$$

Let the mapped coordinates $i\beta$'s corresponding to the points given by Eq. (3) be $\pm i\beta_1$, $\pm i\beta_2$, $\pm i\beta_3$, $\pm i\beta_4$ $(0<\beta_1<\beta_2<\beta_3<\beta_4)$. Under this circumstance, mapping (1) $[-\pi<\beta\leq\pi]$ is unique (one-to-one correspondence) except for $\alpha=0$.

2.2 Basic governing equations

Fluid enclosed is assumed to be substantially incompressible and Boussinesq approximation is introduced. As thermal boundary conditions, the cavity wall, the left cylinder wall, and the right cylinder wall are assumed to be kept at constant temperatures T_0, $T_L(>T_0)$, and T_R respectively. Hereafter dimensionless temperature T is introduced as (local temperature$-T_0$)$/(T_L-T_0)$. Grashof number G_r, which will appear, is based on the temperature difference (T_L-T_0) and reference length L such that (the distance between the centers of the cylinders)$/L\equiv b-a$. Then the dimensionless equation of vorticity transport is given by

$$J\frac{\partial\zeta}{\partial t}+\frac{\partial(\zeta,\psi)}{\partial(\alpha,\beta)}=\frac{1}{\sqrt{G_r}}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\zeta+\frac{\partial(T, y)}{\partial(\alpha,\beta)} \tag{4}$$

$$J\zeta+\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)\psi=0 \tag{5}$$

$$J=\frac{\partial(x, y)}{\partial(\alpha,\beta)} \tag{6}$$

where ζ=vorticity, ψ=stream function. The energy equation is expressed, if dissipation terms are neglected, by

$$J\frac{\partial T}{\partial t}+\frac{\partial(T,\psi)}{\partial(\alpha,\beta)}=\frac{1}{P_r\sqrt{G_r}}\left(\frac{\partial^2}{\partial\alpha^2}+\frac{\partial^2}{\partial\beta^2}\right)T \quad (7)$$

where P_r stands for a Prandtl number.

2.3 Boundary conditions

Along $\alpha=\alpha_\infty$ (cavity wall)

$$\psi=0,\ \frac{\partial\psi}{\partial\alpha}=0,\ T=0 \quad (8)$$

In the following for any scalar quantity $F(\alpha,\beta)$, F_c and F_s are introduced as

$$F_c(\alpha,\beta)\equiv\frac{1}{2}\{F(\alpha,\beta)+F(\alpha,-\beta)\}$$

$$F_c(\alpha,\beta)\equiv\frac{1}{2}\{F(\alpha,\beta)-F(\alpha,-\beta)\}$$

Along the line section $\Im(z)=0$ and $\alpha=0$, for any scalar quantity ϕ the following hold:

$$\phi(0,\beta)=\phi(0,-\beta) \quad (9)$$

$$\frac{\partial}{\partial\alpha}\phi(0,\beta)=-\frac{\partial}{\partial\alpha}\phi(0,-\beta) \quad (10)$$

Thus, along $\alpha=0$, $0<\beta<\beta_1$ or $\beta_2<\beta<\beta_3$ or $\beta_4<\beta<\pi$

$$\frac{\partial}{\partial\alpha}T_c=0,\ T_s=0 \quad (11)$$

$$\frac{\partial}{\partial\alpha}\psi_c=0,\ \psi_s=0 \quad (12)$$

$$\frac{\partial}{\partial\alpha}\zeta_c=0,\ \zeta_s=0 \quad (13)$$

Along $\alpha=0$, $\beta_1<\beta<\beta_2$ (left cylinder surface)

$$T_c=0,\ T_s=0 \quad (14)$$

$$\psi_c=C_1(\text{constant}),\ \psi_s=0 \quad (15)$$

$$\frac{\partial}{\partial\alpha}\psi_c=0,\ \frac{\partial}{\partial\alpha}\psi_s=0 \quad (16)$$

Along $\alpha=0$, $\beta_3<\beta<\beta_4$ (right cylinder surface)

$$T_c=\frac{T_R-T_0}{T_L-T_0},\ T_s=0 \quad (17)$$

$$\psi_c=C_1\ (\text{constant}),\ \psi_s=0 \quad (18)$$

$$\frac{\partial}{\partial\alpha}\psi_c=0,\ \frac{\partial}{\partial\alpha}\psi_s=0 \quad (19)$$

2.4 Multiplicity of the domain

Under a conformal coordinate system, the condition for a multiply-connected region is given by

$$\oint\frac{\partial\zeta}{\partial\alpha}d\beta=0 \quad (20)$$

integrated along each closed boundary (α=constant) if each surface is stationary and kept at a constant temperature for substantially incompressible Newtonian fluids, which is the case, since $\oint(\partial p/\partial\beta)\,d\beta=0$, p: pressure (scalar quantity).

3. Spectral Formulation

Fourier spectral formulation is adopted, that is,

$$\psi=\sum_{n=1}^{\infty}\psi_{sn}(\alpha,t)\sin n\beta+\sum_{n=0}^{\infty}\psi_{cn}(\alpha,t)\cos n\beta \quad (21)$$

$$\zeta=\sum_{n=1}^{\infty}\zeta_{sn}(\alpha,t)\sin n\beta+\sum_{n=0}^{\infty}\zeta_{cn}(\alpha,t)\cos n\beta \quad (22)$$

$$T=\sum_{n=1}^{\infty}T_{sn}(\alpha,t)\sin n\beta+\sum_{n=0}^{\infty}T_{cn}(\alpha,t)\cos n\beta \quad (23)$$

Eq. (16) can be replaced by

$$\zeta_c(h,\beta)+\frac{2}{h^2}\{\psi_c(h,\beta)-C_1\}/J=0 \quad (24)$$

$$\zeta_s(h,\beta)+\frac{2}{h^2}\psi_s(h,\beta)/J=0 \quad (25)$$

h: coordinate of α adjacent to $\alpha=0$. Similarly Eq. (19) can be replaced by

$$\zeta_c(h,\beta)+\frac{2}{h^2}\{\psi_c(h,\beta)-C_2\}/J=0 \quad (26)$$

$$\zeta_s(h,\beta)+\frac{2}{h^2}\psi_s(h,\beta)/J=0 \quad (27)$$

A treatise on a combined boundary condition along $\alpha=0$ is the same as in Mochimaru and Bae (2003). Multiple connection gives rise through Eqs. (5) and (19) to C_1 and C_2 in a finite difference approximation

$$\int_\Omega\left(\frac{\partial}{\partial\alpha}\frac{1}{J}\right)\frac{1}{2h^2}\{8\psi(h,\beta)-\psi(sh,\beta)-7C\}\,d\beta + \int_\Omega\frac{1}{J}\frac{3}{sh^3}\{\psi(2h,\beta)-4\psi(h,\beta)+3C\}\,d\beta=0 \quad (28)$$

where Ω means the cylinder surface (left or right) and for the left $C=C_1$ and for the right $C=C_2$. Expecially at the points $\alpha=0$, $\beta=\beta_i(z=z_i)$, $i=1, \cdots, 4$,

$$\frac{\partial}{\partial\alpha}\left(\frac{1}{J}\right)=2\left|\frac{d\alpha}{d\zeta^*}\right|\left|\frac{d^2\zeta^*}{dz^2}\right| \tag{29}$$

$$\frac{1}{J}=0 \tag{30}$$

$$\left|\frac{d\alpha}{d\zeta^*}\right|=\frac{\cosh\alpha_0}{\sin\beta}\left|\frac{2}{(1-i\,sn)^2}\right| \times(1-sn^2)^{1/2}(1-k^2sn^2)^{1/2} \tag{31}$$

$$i\,sn=\frac{\cos\beta-\cosh\alpha_\infty}{\cosh\alpha_\infty+\cos\beta} \tag{32}$$

$$\left|\frac{d^2\zeta^*}{dz^2}\right|=\left|\frac{2c^2}{(z-a)^3}+\frac{2d^2}{(z-b)^3}\right| \tag{33}$$

4. Numerical Integration Scheme

Spatial discretion is based on a finite difference scheme with a possibility of non-uniform grid spacing in α, whereas time discretion is based on a forward approximation, where actually n-th coordinate α_n in α is given, with a suitable parameter γ, by

$$\alpha_n=h\left\{\frac{\sinh(n-1)\gamma}{\sinh\gamma}+1\right\} \tag{34}$$

which results in $\alpha_2=2h$, $\alpha_1=h$. With a stationary uniform field at first, semi-implicit time integration algorithm, independent of each Fourier component, is applied to get a steady-state solution, where to get a higher numerical stability local acceleration terms is modified as a diagonal dominant form. Mean Nusselt number Nu_m on the left cylinder based on the temperature difference T_L-T_0 and length L is given by

$$Nu_m=-\frac{1}{L_0}\left(\int_{\beta_1}^{\beta_2}\frac{\partial T}{\partial\alpha}d\beta+\int_{-\beta_2}^{-\beta_1}\frac{\partial T}{\partial\alpha}d\beta\right) =-\frac{2}{L_0}\int_{\beta_1}^{\beta_2}\frac{\partial T_c}{\partial\alpha}d\beta \tag{35}$$

where L_0 is a perimeter of the left cylinder, and

$$L_0=2\int_{\beta_1}^{\beta_2}\sqrt{J(0,\beta)}\,d\beta \approx 2\pi\Big/\sqrt{1-\left(\frac{d}{a-b}\right)^2} \tag{63}$$

5. Numerical Results and Discussion

Steady-state fields (streamlines and isotherms) for a configuration corresponding to $\alpha_\infty=1.762$, $a=0.7$, $b=1.0$, $c=0.07$, $d=0.05$ and $k=0.7$ ($z_2-z_1\approx 0.1424$, $z_4-z_3\approx0.1032$) are given in Figs. 1 and 2 at $G_r=10^5$, $P_r=0.7$ (air), $(T_R-T_0)/(T_L-T_0)=0$, in Figs. 3 and 4 at $G_r=10^3$, $P_r=0.7$ (air),

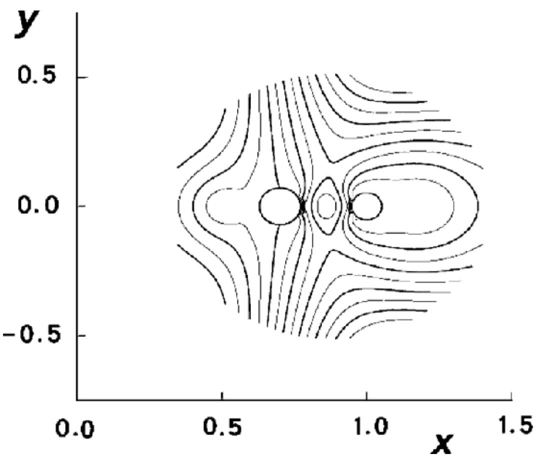

Fig. 1 Steady-state streamlines ($G_r=10^5$, $P_r=0.7$, $\delta\psi$ (difference of the level of ψ) $=5\times10^{-7}$ and $N_m=5.91$)

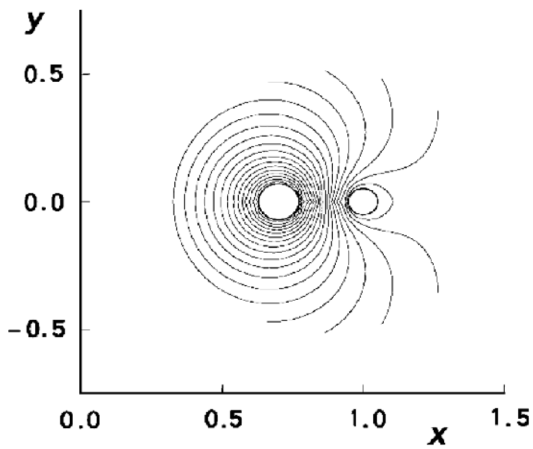

Fig. 2 Steady-state isotherms ($G_r=10^5$, $P_r=0.7$, $\delta T=0.05$ and $Nu_m=5.91$)

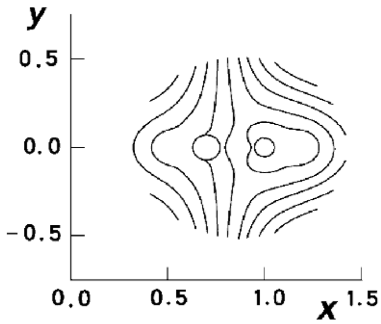

Fig. 3 Steady-state streamlines ($G_r=10^5$, $P_r=0.7$, $\delta\psi=10^{-7}$ and $Nu_m=4.68$)

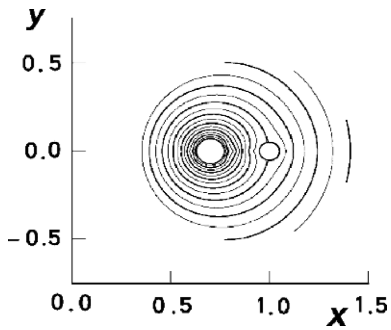

Fig. 4 Steady-state isotherms ($G_r=10^3$, $P_r=0.7$, $\delta T=0.05$ and $Nu_m=4.68$)

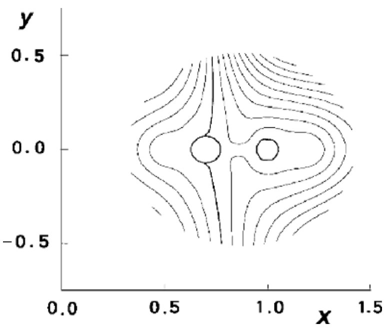

Fig. 5 Steady-state streamlines ($G_r=1.5\times10^7$, $P_r=0.7$ and $\delta\psi=2.5\times10^{-5}$)

$(T_R-T_0)/(T_L-T_0)=0.5$, and in Figs. 5 and 6 at $G_r=1.5\times10^7$, $P_r=0.7$ (air), $(T_R-T_0)/(T_L-T_0)=0.5$. The pattern of streamlines (macroscopically one counterclockwise circulation and a clockwise one) is vastly different from those in infinite extension, whereas the thermal fields are nearly the same as those in pure heat conduction in a cavity as long as G_r is less than a certain value

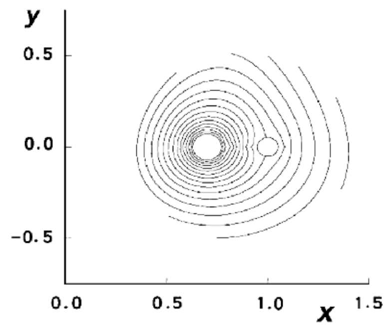

Fig. 6 Steady-state isotherms ($G_r=1.5\times10^7$, $P_r=0.7$, $\delta T=0.05$ and $Nu_m=4.73$)

of order 10^5 for the current configuration.

6. Conclusions

Focus is placed on the usefulness to the possibility of application of a spectral finite difference scheme to a multiply-connected two-dimensional region. In this paper, triply-connected region is considered. As a result, steady-state natural convection heat transfer characteristics from cylinders in a multiply-connected bounded region are clarified, using a spectral finite difference scheme. Current proposed spectral finite difference scheme for a multiply-connected region can be applied in principle for two-dimensional laminar natural convection, namely $G_r<2\times10^7$ for $P_r=0.7$ based on the reference length given, which is supposed from the availability of steady-state solutions.

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